

Determination of the stability limit of a thermoacoustic engine by means of finite elements

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Introduction

Thermoacoustic (TA) heat engines are engines which convert heat into acoustic power. This relatively new class of heat engines gains commercial interest, because of their low number of moving parts and high potential efficiency [1]. Especially traveling wave thermoacoustic engines are capable of reaching this high efficiency, because traveling wave engines perform a thermodynamic cycle similar to the Stirling engine cycle [4].

In thermoacoustic systems, the time-averaged effects induced by a strong acoustic field are of primary interest. For example, the thermoacoustic power flow in a tube is the time-average of the oscillatory pressure and volume flow [10]. The generation of this power flow is done by consuming heat. This heat consumption has an influence on the mean temperature field. To model the coupled effects of acoustics and energy transfer, the linear acoustics assumptions have to be relaxed and a finite-amplitude approach is required. Since 3D CFD simulations generally require too much computational cost to serve as a design tool, in literature often a perturbation approach is chosen including the assumption of time-harmonic oscillations. The model obtained after doing these assumptions is incorporated in the code DeltaEC [13], which is nowadays a well-known design tool for thermoacoustic systems. The theory behind DeltaEC includes geometric restrictions to parts, because it is based on the assumption that the acoustic wave has a significant propagation direction. This is indeed the case in tubings where the cross-section length scale is much smaller than the wavelength (also called the low reduced frequency assumption [11]). This assumption does not hold at compliances, t-joints and other more complex shapes. To quantify thermoacoustic effects in such shapes, a three-dimensional model is required.

In this paper, a linear 3D thermoacoustic model is shown. With the linear model, no finite-amplitude effects can be predicted. Therefore, the use is somewhat restricted. For thermoacoustic engines, we can apply the model to the numerical computation of the onset criterion, called the stability limit. The stability limit is expressed in an amount of heat, or a temperature difference in the engine, for which neutrally stable (undamped) oscillations can exist. This temperature or heat is called the onset temperature, or the onset heat input respectively. The neutrally stable point is always present for a self-starting thermoacoustic engine (otherwise it would not be self-starting). After crossing the stability limit, a minor disturbance, which is always present, will be amplified until an amplitude is reached at which

nonlinear effects will bound further growth. To compute the final saturation amplitude, a nonlinear model is required. In the future, the linear model presented in this paper will be used as the first expansion of a second order model.

In the next section, the model to calculate the stability limit is explained in more detail. After that, an example 2D standing wave engine is used to demonstrate the model. For this model, a 2D implementation in the commercial finite element package Comsol Multiphysics is used. A comparison is made with existing 1D theory.

Model

In this section, an overview of the model and methodology is given with which the stability limit of a thermoacoustic engine can be calculated. The onset temperature at the stability limit corresponds to a temperature distribution in the engine for which the thermoacoustic eigenmode with the lowest onset temperature becomes neutrally stable. This means that when a disturbance is present, it neither increases nor decreases in amplitude. In most cases, the only eigenmode which will become unstable is the first one. This is true because the higher modes generally have an unfavorable impedance in the regenerator or stack. However, in literature some cases exist where a higher mode becomes unstable at an earlier stage [14]. The methodology takes two steps, which are executed iteratively. First, we apply a certain amount of heat to the system. We assume mechanical equilibrium and body forces are neglected. In steady state this will generate a non-uniform temperature field. This temperature field can be obtained by solving the steady state heat equation:

$$\nabla \cdot (\kappa_m \nabla T_m) = 0, \quad (1)$$

where κ_m denotes the mean (temperature dependent) thermal conductivity of the fluid and T_m the mean temperature. $T_m = T_w$ at a constant mean temperature boundary Γ_w and $-\kappa_m \nabla T_m \cdot \mathbf{n} = q_{out}$ at a heat flux boundary. When the steady state heat equation is solved, a mean temperature field is obtained, from which the mean density field can be computed using the perfect gas law, because the mean pressure is constant throughout the domain due to the assumed mechanical equilibrium. The next step is to look at the stability of the thermoacoustic eigenmode of interest. The thermoacoustic field can be described by a linearization of the governing equations of the fluid (continuity, momentum and energy equation). We will consider only Fourier-Newtonian perfect gases. Thermoacoustic effects are included when

the mean temperature and density fields are assumed to be non-constant. We assume sinusoidal perturbations and adopt Swifts phasor notation using complex exponentials [9]: $\xi'(\mathbf{x}, t) = \Re(\xi_1 e^{i\omega t})$. Where ξ' is one of the perturbed (acoustic) quantities: density ($\rho' = \rho - \rho_m$), pressure ($p' = p - p_m$), temperature ($T' = T - T_m$) or velocity ($\mathbf{u}' = \mathbf{u}$). Substituting these expressions in the governing equations, linearization, eliminating the linearized perfect gas law and applying the phasor notation yields the following equations, known as the Linearized Navier Stokes Fourier equations (LNSF):

$$i\omega \left(\frac{p_1}{p_m} - \frac{T_1}{T_m} \right) + \nabla \cdot \mathbf{u}_1 - \underbrace{\frac{\mathbf{u}_1 \cdot \nabla T_m}{T_m}}_{(1)} = 0, \quad (2a)$$

$$i\omega \rho_m \mathbf{u}_1 + \nabla p_1 = \nabla \cdot \boldsymbol{\tau}_1, \quad (2b)$$

$$\rho_m c_p (i\omega T_1 + \underbrace{\mathbf{u}_1 \cdot \nabla T_m}_{(2)}) - i\omega p_1 = -\nabla \cdot \mathbf{q}_1, \quad (2c)$$

where p_1 denotes the acoustic pressure phasor, $\mathbf{u}_1$ the acoustic velocity phasor and T_1 the acoustic temperature fluctuation phasor. Subscript m denotes mean (time-invariant) quantities. Note that the gas law has been used to replace the gradient of the density with the gradient of the temperature. $\mathbf{q}_1$ is the fluctuating heat flux ($-\kappa_m \nabla T_1$) and $\boldsymbol{\tau}_1$ denotes the acoustic viscous stress tensor, defined as

$$\boldsymbol{\tau}_1 = \mu_m [\nabla \mathbf{u}_1 + (\nabla \mathbf{u}_1)^T] + (\mu_B - \frac{2}{3}\mu_m) [\nabla \cdot \mathbf{u}_1] \mathbf{I}, \quad (3)$$

where μ_m denotes the mean dynamic viscosity, μ_B the mean bulk viscosity and $\mathbf{I}$ the unit tensor [6]. The terms in equations (2) underbraced with (1) and (2) are due to the inhomogeneous mean temperature field. Assuming again a homogeneous mean temperature, the equations reduce to the model already derived by Pierce [7].

The stability of an eigenmode can be determined from the imaginary part of the eigenfrequency [5]. At the stability limit, one eigenfrequency has an imaginary part equal to zero, while the other eigenfrequencies all have $\Im(\omega_n) > 0$. To find the onset temperature, the finite element computations are parametrized with two variables: the hot heat exchanger temperature or heat input and the oscillation frequency. For an arbitrary temperature distribution and frequency, not all boundary conditions can be satisfied. The problem now is to find a combination of $\vec{x} = (T_w, \omega)$, or (q_{in}, ω) for which all known boundary conditions are met. An error function $\vec{F}$ is created which describes the difference between the calculated boundary condition values and the objective boundary conditions. Since generally the boundary conditions of an acoustic system can be nonlinearly dependent on the frequency, a nonlinear search method is chosen to find the acoustic eigenmode. An iterative solution algorithm is used to find the corresponding hot heat exchanger temperature and frequency for the stability limit. This method is called Broyden's method [3] and is a multidimensional variant of Secant's method. The updating of the solution vector follows the strategy of Newton's method:

$$\vec{x}_{n+1} = x_n - \mathbf{J}_n^{-1} \vec{F}_n \quad (4)$$

In this case $\mathbf{J}_n$ is an approximate Jacobian at iteration n which is computed by

$$\mathbf{J}_n = \mathbf{J}_{n-1} + \frac{\Delta \vec{F}_n - \mathbf{J}_{n-1} \Delta \vec{x}_n}{\|\Delta \vec{x}_n\|^2} \Delta \vec{x}_n^T, \quad (5)$$

with $\Delta \vec{F}_n = \vec{F}_n - \vec{F}_{n-1}$ and $\Delta \vec{x}_n = \vec{x}_n - \vec{x}_{n-1}$.

Example: 2D standing wave engine

An example of an application is shown in figure 1. It shows a 2D standing wave engine. The dimensions are listed in table 1. When sufficient heat is added to the hot heat exchangers (the left and red part), the first (thermo)acoustic eigenmode becomes unstable. The acoustic medium is air, so a 3D variant of this engine will typically run with a dominating oscillation frequency close to $c_0/4L_0 \approx 500$ Hz ($\lambda/4$ resonator). Where c_0 is the ambient speed of sound (≈ 340 m/s) at the reference temperature (20°C) and L_0 is the length of the engine.

The sound is radiated from the right side (acoustic

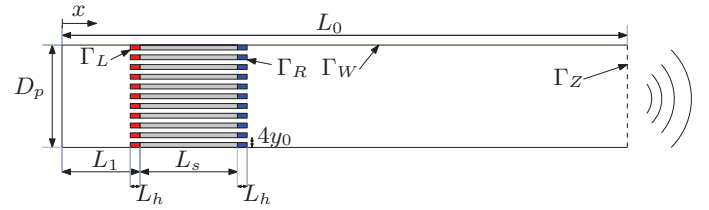


Figure 1: Schematic of the 2D standing wave engine including dimensional parameters

Param.	Value	Param.	Value
L_0	17 cm	L_1	3 cm
L_s	1.5 cm	L_h	0.15 cm
W_0	2.1 cm	y_0	0.5 mm
ϕ	10/21		

Table 1: Dimensional parameters of the 2D standing wave engine

open end), where a radiation impedance of a baffled piston is applied [2]. At all other walls, the no-slip boundary condition is applied for the acoustic velocity. The acoustic temperature fluctuation is assumed to be zero at the walls. An overview of the boundary conditions is given in table 2. The mean pressure is assumed to be atmospheric (101325 Pa), and the T_R is set to room temperature (293.15 K). The hot heat exchanger temperature for onset is unknown and will be obtained as part of the solution. The starting value for the solving method is chosen as 500 Hz. A guess onset temperature is obtained with the short stack approximation [8] and is $\approx 230^\circ\text{C}$.

Iteratively, the error between the computed impedance (z_{FE}) at the open end and the required radiation impedance (z_p) is made zero, so the error function is $\vec{F} = (\Re(z_p - z_{FE}), \Im(z_p - z_{FE}))$. The Broyden solver changes the frequency and the hot heat exchanger temperature (the two unknowns) until the error is smaller than a given norm. Moreover, a convergence

Quantity	Boundary and b.c.	
	Boundary	Condition
Mean temp.	Γ_L	$T_m = T_L$
	Γ_R	$T_m = T_R$
	Γ_W	$q_{out} = 0$
	Γ_Z	$q_{out} = 0$
Velocity	Γ_L	$\mathbf{u}_1 = \mathbf{0}$
	Γ_R	$\mathbf{u}_1 = \mathbf{0}$
	Γ_W	$\mathbf{u}_1 = \mathbf{0}$
	Γ_Z	free
Ac. temp.	Γ_L	$T_1 = 0$
	Γ_R	$T_1 = 0$
	Γ_W	$T_1 = 0$
	Γ_Z	$\mathbf{q}_1 \cdot \mathbf{n} = 0$
Ac. press.	Γ_L	free
	Γ_R	free
	Γ_W	free
	Γ_Z	$p_1 = 1\text{Pa}$

Table 2: Boundary conditions for the 2D standing wave engine

norm is set to the change in the solution variables: $\Delta(T_{Ln}, \omega_n) < (10^{-3} \text{ rad/s}, 10^{-3} \text{ K})$. The error in the solution variables determines the required amount of iterations. A mesh study has been performed to determine the required mesh size. The final mesh, comprising 71590 elements is chosen such that the results become satisfactory mesh-independent.

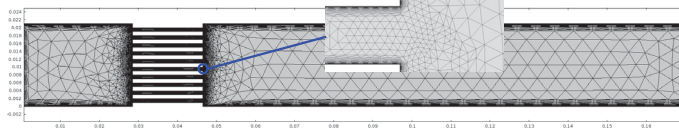


Figure 2: Final mesh used of the 2D standing wave engine

Results

The obtained results comprise the nodal solution of the mean temperature and density field and the acoustic fluctuation phasors. The phasors are scaled such that the acoustic pressure at the radiation impedance boundary equals 1 Pa. A close-up of the acoustic temperature $\|T_1\|$ and x-velocity $\|u_1\|$ are shown in figure 3.

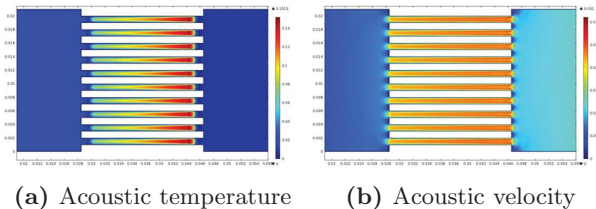


Figure 3: Close-up of the magnitude of the acoustic temperature and velocity in the stack.

The geometry of the shown example is chosen such that the low-reduced frequency assumption [11] is valid as well. Therefore, to verify the implementation of the finite element model, a comparison has been made. With both models, the stability limit has been calculated. Applying the low reduced frequency assumptions to the LNSF

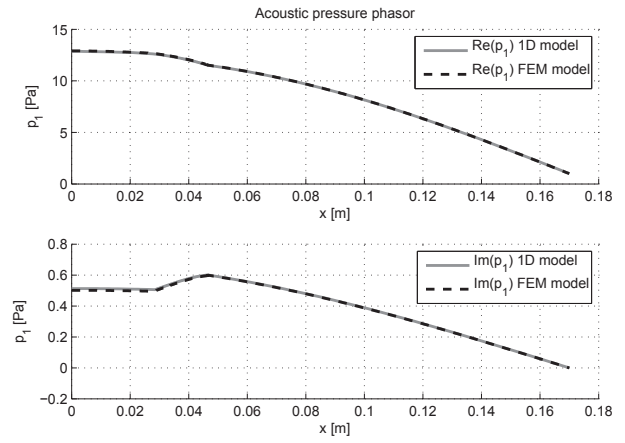


Figure 4: Comparison of the area-averaged acoustic pressure of the FEM model with the 1D model.

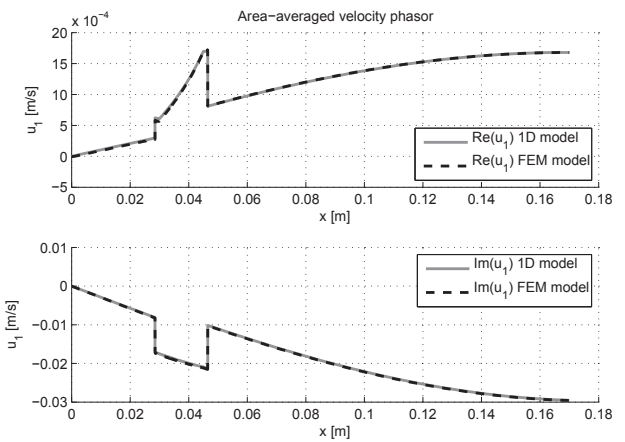


Figure 5: Comparison of the area-averaged acoustic velocity of the FEM model with the 1D model.

equations, an effectively 1D model is obtained [10]:

$$\frac{dp_1}{dx} = \frac{i\omega\rho_m}{(1-f_\nu)} \langle u_1 \rangle, \quad (6)$$

$$\frac{d\langle u_1 \rangle}{dx} = \frac{i\omega(1+(\gamma-1))f_\kappa}{\gamma p_m} p_1 + \frac{(f_\kappa - f_\nu)}{(1-f_\nu)(1-\text{Pr})} \frac{d\langle T_m \rangle}{dx} \langle u_1 \rangle, \quad (7)$$

where f_x are the viscous (ν) and thermal (κ) Rott functions [10], Pr, and the Prandtl number and u_1 the acoustic x-velocity phasor. $\langle \rangle$ denotes area-averaging over the fluid-filled cross-sectional area, i.e. $\langle u_1 \rangle = \frac{1}{S_f} \int_{S_f} u_1 dS$. At the discontinuities between the heat exchangers and the hot space and resonator, continuity of p_1 and $\phi \langle u_1 \rangle$ is assumed, where ϕ denotes the porosity (1 at the hot and cold side, 10/21 in the stack and heat exchangers). The method to obtain the stability limit closely follows the steps as explained by Ueda et al. [12], but with the important difference that the mean temperature field is not imposed, but computed from the steady heat equations and the temperature boundary conditions. The steady heat equation can be treated in a similar way. Therefore, the assumption has to be done that the pore aspect ratio is large. Applying this assumption on the heat equation and taking a first order

perturbation expansion yields the following 1D model:

$$\frac{d}{dx} \left(-\kappa_m \frac{d\langle T_m \rangle}{dx} \right) = \frac{h}{2y_0} (T_w - \langle T_m \rangle), \quad (8)$$

where T_w denotes the wall temperature. The derived heat transfer coefficient for conduction between two parallel plates with a separation distance of $2y_0$, equals $h = 6\kappa_m/y_0$. For the hot duct, the stack and the resonator, the wall temperature is set equal to the mean fluid temperature, such that no heat input is applied in these segments. At the interfaces between the heat exchangers and the hot duct and resonator, continuity of the mean temperature ($\langle T_m \rangle$) and heat flow ($-\kappa_m \phi \frac{dT_m}{dx}$) is assumed. Equations (6-8) are solved using a fourth order Runge-Kutta finite difference scheme. To compare results of the finite element model with the 1D model, the finite element solution is area-averaged over the fluid-filled area of the engine as well, but this integration is done numerically. Figures 4-5 show a comparison of the acoustic pressure and area-averaged velocity profile respectively at the onset conditions. The predicted onset temperature of the finite element model is 258°C at 501 Hz. The 1D model results a slightly lower onset temperature of 252°C at 502 Hz. The temperature profiles are compared in figure 6. The slightly lower onset temperature of the 1D model can be explained by the fact that some viscothermal losses incorporated in the FEM model are not taken into account in the 1D model. A small difference is visible in the normalised intensity plot (figure 7) in the hot part. This is due to end cap thermal relaxation loss [9], which is not taken into account in the 1D model.

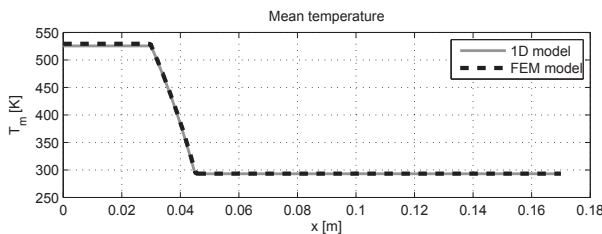


Figure 6: Comparison of mean temperature at the stability limit of the FEM model with the 1D model.

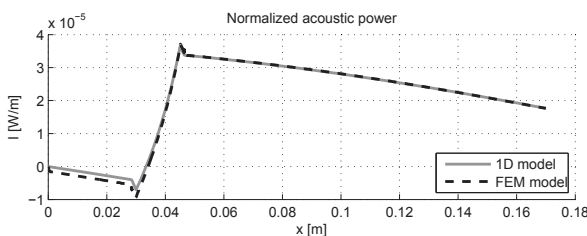


Figure 7: Comparison of normalized acoustic power at the stability limit of the FEM model with the 1D model.

Conclusions

A finite element formulation of the LNSF equations has been successfully implemented and used to find the stability limit of a 2D standing wave engine. The results have been verified using existing 1D theory and a good

agreement has been found. Future work will involve the expansion of the variables up to second order, to include weakly nonlinear effects, such as amplitude saturation mechanisms and second order acoustic streaming.

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